ON INEQUALITIES FOR
DIFFERENTIAL OPERATORS

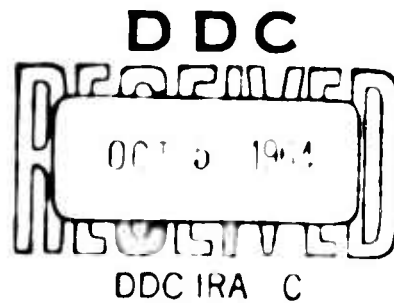Richard Bellman

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SUMMARY

In this paper ~~we study~~ the following problem: Given that certain functionals of u and its derivatives belong to given L—classes over the infinite interval, what can be said about the L—classes of other functionals? Utilizing a simple device from the theory of linear differential equations, ~~we obtain~~ ~~several~~ results, due to Landau, Kolmogoroff, Halperin—von Neumann, and Nagy, together with some extensions.

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In this paper we study the following problem: Given that certain functionals of u and its derivatives belong to given L-classes over the infinite interval, what can be said about the L-classes of other functionals? Utilizing a simple device from the theory of linear differential equations, we obtain a number of results due to Landau, Kolmogoroff, Halperin-von Neumann, and Nagy, together with some extensions.

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1. INTRODUCTION

Since the original result of Hadamard concerning a relation between bounds for $u(t)$, $u'(t)$ and $u''(t)$ for t in a finite interval, a great deal of work has been done in the field of inequalities relating various functionals of u and its derivatives over finite and infinite intervals. A discussion of some questions of this nature may be found in the book by Hardy, Littlewood and Polya, [4], and a collection of more recent results in the appendix to the Russian edition, edited by V. Levin, [8]. For the case where $u(t)$ is an ordinary polynomial, a trigonometric polynomial, or an entire function of exponential type, we refer to the book by Boas, [3].

In this paper we are interested in problems of the following type: Given that certain functionals of u and its derivatives belong to given L-classes over the infinite interval, what can be said about the L-classes of other functionals? Utilizing a method originally sketched in our book on stability theory, [2], p. 117, we derive in a uniform fashion a number of results previously obtained by Landau, [7], Kolmogoroff, [6], Halperin-von Neumann, [5], and Nagy, [9], together with some extensions.

Although we do not examine the deeper problem of determining best possible bounds, we shall indicate briefly how the methods we present can be used to obtain inequalities as well

as inclusion results.

2. NOTATION

We shall write $u \in L^p$ for the statement that $\int_0^\infty |u|^p dt < \infty$. When the interval is $[-\infty, \infty]$ instead of $[0, \infty]$, we shall write $u \in L^p[-\infty, \infty]$. The notation $u \in L^p$ with $p = \infty$ means that $|u| \leq c_1 < \infty$ for $0 \leq t < \infty$, or $-\infty < t < \infty$, depending upon the interval under consideration.

3. SOME INEQUALITIES CONNECTING u, u' AND u"

In this section we shall consider inequalities of the type presented by Hadamard, Esclangon, Landau, [7], and Kolmogoroff, [6]. The paper by Kolmogoroff contains an account of best possible constants.

We wish to establish the following result:

Theorem 1. If $u \in L^p$, $p \geq 1$, $u'' \in L^r$, $r \geq 1$, then $u, u' \in L^m$ for $m \geq \max [p, r]$.

The two most interesting cases of this result, $p = r = m = \infty$, and $p = r = m = 2$, were treated in [2].

Proof. Write

(1) $$u'' - u = f + g,$$

where $f \in L^r$ and $g \in L^p$. Considering this relation to be a linear differential equation for u, with forcing term $f + g$, we can write u in the form

(2) $$u = c_1 e^t + c_2 e^{-t} + \frac{1}{2} \int_0^t \left[e^{(t-s)} - e^{-(t-s)} \right] \left[f(s) + g(s) \right] ds.$$

The assumption that $f \in L^r$ and $g \in L^p$ implies that the integral $\int_0^\infty e^{-s}[f(s) + g(s)]ds$ converges and that $e^{-t} \int_0^t e^s[f(s) + g(s)]ds$ is bounded as $t \to \infty$. Hence in order that $u \in L^p$, it is necessary that

(3) $\qquad c_1 + \frac{1}{2}\int_0^\infty e^{-s}[f(s) + g(s)]ds = 0.$

Using this relation, (2) takes the form

(4) $\qquad u = c_2 e^{-t} + \frac{e^t}{2}\int_t^\infty e^{-s}[f + g]ds - \frac{e^{-t}}{2}\int_0^t e^s[f + g]ds,$

which yields

(5) $\qquad u' = -c_2 e^{-t} + \frac{e^t}{2}\int_t^\infty e^{-s}[f + g]ds + \frac{e^{-t}}{2}\int_0^t e^s[f + g]ds.$

From this representation, we readily derive the stated results. To begin with, consider the case where $m = \infty$. Since, using Holder's inequality,

$$\left| \int_t^\infty e^{-s}f\,ds \right| \leq \left(\int_t^\infty |f|^p ds \right)^{\frac{1}{p}} \left(\int_t^\infty e^{-qs} ds \right)^{\frac{1}{q}},$$

$$\leq \frac{e^{-t}}{q} \left(\int_0^\infty |f|^p ds \right)^{\frac{1}{p}},$$

(6)

$$\left| \int_0^t e^s f\,ds \right| \leq \left(\int_0^t e^{qs} ds \right)^{\frac{1}{q}} \left(\int_0^t |f|^p ds \right)^{\frac{1}{p}}$$

$$\leq \frac{e^t}{q} \left(\int_0^\infty |f|^p ds \right)^{\frac{1}{p}},$$

with corresponding results for integrals involving g, we see that $|u|$ and $|u'|$ are uniformly bounded for $t \geq 0$.

If $1 \leq m < \infty$, we have

$$|u'(t)|^m \leq c_4(m)\left[e^{-mt} + e^{mt}\left(\int_t^\infty e^{-s}|f|\,ds\right)^m\right.$$

$$+ e^{mt}\left(\int_t^\infty e^{-s}|g|\,ds\right)^m$$

(7)

$$+ e^{-mt}\left(\int_0^t e^s|f|\,ds\right)^m$$

$$\left.+ e^{-mt}\left(\int_0^t e^s|g|\,ds\right)^m\right].$$

Consider a typical term on the right-hand side.

We have

$$\int_0^\infty e^{mt}\left(\int_t^\infty e^{-s}|f|\,ds\right)^m dt$$

(8)

$$\leq \int_0^\infty e^{mt}\left(\int_t^\infty e^{-\frac{qs}{2}}ds\right)^{\frac{m}{q}}\left(\int_t^\infty e^{-\frac{ps}{2}}|f|^p\,ds\right)^{\frac{m}{q}} dt$$

$$\leq c_5 \int_0^\infty e^{\frac{mt}{2}}\left(\int_t^\infty e^{-\frac{ps}{2}}|f|^p\,ds\right)^{\frac{m}{p}} dt.$$

Integrating by parts, we have

$$\int_0^\infty e^{\frac{mt}{2}}\left(\int_t^\infty e^{-\frac{ps}{2}}|f|^p\,ds\right)^{\frac{m}{p}} dt$$

$$= \frac{m}{p}\int_0^\infty e^{\left(\frac{m-p}{2}\right)t}|f|^p\left(\int_t^\infty e^{-\frac{ps}{2}}|f|^p\,ds\right)^{\frac{m}{p}-1} dt$$

(9)

$$\leq c_6 \int_0^\infty |f|^p\left(\int_t^\infty |f|^p\,ds\right)^{\frac{m}{p}-1} dt$$

$$= \frac{c_6 p}{m}\left(\int_0^\infty |f|^p\,ds\right)^{\frac{m}{p}} < \infty.$$

We see that the condition that $m \geq p$ plays an essential role. This completes the proof.

Observe that we have actually established a stronger result than stated. It is not necessary that $u \in L^p$ and $u'' \in L^r$, but only that

$$u = \sum_{k=1}^{N} f_k,$$

(10)

$$u'' = \sum_{k=1}^{N} g_k,$$

with $f_k \in L^{p_k}$, $g_k \in L^{r_k}$, $p_k, r_k \geq 1$, and

(11) $\qquad m \geq \underset{k}{\text{Max}} \; [p_k, r_k].$

4. INEQUALITIES CONNECTING $u, u^{(k)}$ AND $u^{(n)}$.

To obtain a corresponding result for a general triplet of derivatives, u, $u^{(k)}$ and $u^{(n)}$, $n > k > 1$, we use the equation

(1) $\qquad u^{(n)} - u = f + g,$

if $n = 4m + 2$ or odd, and

(2) $\qquad u^{(n)} + u = f + g,$

if $n = 4m$.

The reason for this change of equation lies in our desire to avoid the case where the characteristic equation contains a root with zero real part. The result corresponding to Theorem 1 is

Theorem 2. _If_ $u \in L^p$, $u^{(n)} \in L^r$, p, $r \geq 1$, $n > 1$,
then $u^{(k)} \in L^m$ _for_ $m \geq \text{Max}[p,r]$ and $k = 0, 1, 2, \ldots, n - 1$.

The proof follows the same lines as above.

5. ALTERNATIVE APPROACH FOR u, u' AND u''

If in place of the statement of Theorem 1 we wish to
obtain actual inequalities, we can proceed as follows. From
the relation

$$(1) \qquad \frac{d}{dt}\left[e^{-t}(u' + u)\right] = e^{-t}(u'' - u),$$

we obtain for $u(t)$ satisfying the hypothesis of Theorem 1
the relation

$$(2) \qquad u' = -u - e^t \int_t^\infty e^{-s}[u'' - u]\,ds.$$

This result is also obtainable from (3.5) upon integrating
by parts and observing that $c_2 = [u(0) - u'(0)]/2$.

Hence

$$(3) \qquad \text{Max}_{t \geq 0} |u'| \leq 2 \,\text{Max}_{t \geq 0} |u| + \text{Max}_{t \geq 0} |u''|.$$

Replacing $u(t)$ by $u(rt)$ for $r > 0$, we obtain the relation

$$(4) \qquad r \,\text{Max}_{t \geq 0} |u'| \leq 2 \,\text{Max}_{t \geq 0} |u| + r^2 \,\text{Max}_{t \geq 0} |u''|,$$

for $r > 0$. From this it follows that

$$(5) \qquad \left(\text{Max}_{t \geq 0} |u'|\right)^2 \leq 8\left(\text{Max}_{t \geq 0} |u|\right)\left(\text{Max}_{t \geq 0} |u''|\right).$$

Similarly, we can obtain an inequality connecting

$\int_0^{\infty} |u'|^m dt$, $\int_0^{\infty} |u|^p dt$ and $\int_0^{\infty} |u''|^r dt$; cf. the arguments

given in [1].

Perhaps the easiest way to obtain extensions of the equation

in (2) is to use the vector-matrix relation

(6) $\frac{d}{dt}(e^{At}x) = e^{At}(Ax + \dot{x})$,

where x is an n-dimensional vector, or

(7) $x = e^{-At} \int_t^{\infty} e^{As}(Ax + \dot{x})ds$,

for A is a stability matrix.

Choosing A suitably, and

(8) $x = \begin{pmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{pmatrix}$,

we obtain a variety of relations connecting $u^{(k)}$ with linear

combinations of u and the derivatives of u. In this way,

we can derive a number of extensions of (5).

6.   AN INEQUALITY OF HALPERIN AND VON NEUMANN AND EXTENSIONS

An extension of the foregoing techniques yields an exten-

sion of the previous results. Let us begin by establishing

Theorem 3. If

a. $u'' + a_1(t)u' + a_2(t)u \in L^p$

(1) b. $u \in L^r$

c. $|a_1(t)|, |a_2(t)| \le c_1 < \infty, \quad 0 \le t,$

then $u, u' \in L^m$ for $m \ge \mathrm{Max}\ (r,p)$.

Proof. Let us discuss only the case where $m = \infty$, which means that we are studying the uniform boundedness of $|u'|$. The general case can be treated in the same fashion. In addition to the device we have been using in the previous sections, we must introduce an additional one.

We write

$$u'' - M^2 u = u'' + a_1(t)u' + a_2(t)u$$

(2)
$$- [a_1(t)u' + a_2(t)u + M^2 u]$$

$$= f(t) - a_1(t)u',$$

where $f(t) \in B[0,\infty]$, by virtue of the assumptions in (1).

Solving for u, we write

$$u = c_1 e^{-Mt} + c_2 e^{Mt} + \frac{1}{2M} \int_0^t \left[e^{M(t-s)} - e^{-M(t-s)} \right] f(s)ds$$

(3)
$$- \frac{1}{2M} \int_0^t \left[e^{M(t-s)} - e^{-M(t-s)} \right] a_1(s)u'(s)ds.$$

Since it is easily established that $[u'' + a_1(t)u' + a_2(t)u] \in B[0,\infty]$, $a_1(t) \in B[0,\infty]$, $i = 1, 2$, imply that $|u'(t)| \le c_3 e^{bt}$ for some constants c_3 and b, we see that

the integrals $\int_0^\infty e^{-Ms} a_1(s) u'(s) ds$ and $\int_0^\infty e^{-Ms} f(s) ds$

converge. Consequently (3) has the form

(4)
$$u = c_1 e^{-Mt} + \frac{1}{2M} \int_0^t e^{-Mt} e^{Ms} f(s) ds - \frac{1}{2M} \int_0^t e^{-Mt} e^{Ms} a_1(s) u'(s) ds$$
$$+ \frac{e^{Mt}}{2M} \int_t^\infty e^{-Ms} [f(s) - a_1(s) u'(s)] ds,$$

and

(5)
$$u' = - M c_1 e^{-Mt} + \frac{1}{2} \int_0^t e^{-Mt} e^{Ms} f(s) ds$$
$$- \frac{1}{2} \int_0^t e^{-Mt} e^{Ms} a_1(s) u'(s) ds$$
$$+ \frac{e^{Mt}}{2} \int_t^\infty e^{-Ms} [f(s) - a_1(s) u'(s)] ds.$$

From this it follows for an appropriate constant c_3 that

(6)
$$|u'| \leq c_2 \left[e^{-Mt} + e^{-Mt} \int_0^t e^{Ms} |f(s)| ds \right.$$
$$+ e^{-Mt} \int_0^t e^{Ms} |u'(s)| ds$$
$$+ e^{Mt} \int_t^\infty e^{-Ms} |f(s)| ds$$
$$\left. + e^{Mt} \int_t^\infty e^{-Ms} |u'(s)| ds \right].$$

Thus, for an appropriate c_3,

(7)
$$\underset{0 \leq t \leq T}{\text{Max}} |u'| \leq c_3 + (\underset{0 \leq t \leq T}{\text{Max}} |u'|)/M$$
$$+ \underset{0 \leq t \leq T}{\text{Max}} (e^{Mt} \int_t^\infty e^{-Ms} |u'(s)| ds),$$

whence

(8)
$$\underset{0 \leq t \leq T}{\text{Max}} |u'| \leq c_3/(1 - 1/M) + \underset{0 \leq t \leq T}{\text{Max}} (e^{Mt} \int_t^\infty e^{-Ms} |u'(s)| ds)/(1 - 1/M)$$

Assuming that $M \geq 3$, we have

(9)
$$\underset{0 \leq t \leq T}{\text{Max}} |u'| \leq c_4 + c_4 \underset{0 \leq t \leq T}{\text{Max}} \left(e^{\frac{Mt}{2}} \int_t^\infty e^{-\frac{Ms}{2}} |u'(s)| ds \right).$$

On the other hand, returning to (6) we have

(10)
$$\int_t^\infty e^{-\frac{Mt}{2}} |u'(t)| dt \leq c_2 \left[\int_t^\infty e^{-\frac{3Mt}{2}} dt \right.$$
$$+ \int_t^\infty e^{-\frac{3Mt}{2}} \left(\int_0^t e^{Ms} |f(s)| ds \right) dt$$
$$+ \int_t^\infty e^{-\frac{3Mt}{2}} \left(\int_0^t e^{Ms} |u'(s)| ds \right) dt$$
$$+ \int_t^\infty e^{\frac{Mt}{2}} \left(\int_t^\infty e^{-Ms} |f(s)| ds \right) dt$$
$$\left. + \int_t^\infty e^{\frac{Mt}{2}} \left(\int_t^\infty e^{-Ms} |u'(s)| ds \right) dt \right].$$

Integrating by parts we obtain

$$\int_t^\infty e^{-\frac{Mt}{2}} |u'(t)| \, dt \leq c_2 \left[\frac{2}{3M} e^{-\frac{3Mt}{2}} \right.$$

$$+ \frac{2}{3M} e^{-\frac{3Mt}{2}} \int_0^t e^{Ms} |f(s)| \, ds$$

$$+ \frac{2}{3M} \int_t^\infty e^{-\frac{Ms}{2}} |f(s)| \, ds$$

$$+ \frac{2}{3M} e^{-\frac{3Mt}{2}} \int_0^t e^{Ms} |u'(s)| \, ds$$

(11)

$$+ \frac{2}{3M} \int_t^\infty e^{-\frac{Ms}{2}} |u'(s)| \, ds$$

$$+ \frac{2e^{\frac{Mt}{2}}}{M} \int_t^\infty e^{-Ms} |f(s)| \, ds$$

$$+ \frac{2}{M} \int_t^\infty e^{-\frac{Mt}{2}} |f(t)| \, dt$$

$$+ \frac{2}{M} e^{\frac{Mt}{2}} \int_t^\infty e^{-Ms} |u'(s)| \, ds$$

$$\left. + \frac{2}{M} \int_t^\infty e^{-\frac{Ms}{2}} |u'(s)| \, ds \right].$$

It follows that

(12) $$e^{\frac{Mt}{2}} \int_t^\infty e^{-\frac{Mt}{2}} |u'(t)| \, dt \leq c_3 \left[1 + \frac{e^{-Mt}}{M} \int_0^t e^{Ms} |u'(s)| \, ds \right].$$

Hence

(13) $$\underset{0 \leq t \leq T}{\text{Max}} \left[e^{\frac{Mt}{2}} \int_t^\infty e^{-\frac{Mt}{2}} |u'(t)| \, dt \right] \leq c_3 + \frac{1}{M} \underset{0 \leq t \leq T}{\text{Max}} |u'(s)|.$$

Combining (8) and (13), we see that $\underset{0 < t < T}{\text{Max}}\ |u'(s)|$ is uniformly bounded for $T \geq 0$. This completes the proof.

The proof of the general result proceeds in analogous fashion.

7. AN INEQUALITY OF NAGY

The following result was obtained by Nagy, [9].

Theorem. _If_ $u \in L^p[-\infty,\infty]$, $p > 0$, $u' \in L^r[-\infty,\infty]$, $r \geq 1$, _then_ $u \in L^{p+b}[-\infty,\infty]$, _for_ $b \geq 0$.

In addition, Nagy obtains a much stronger result with best possible constants. Although we cannot establish, using the previous methods, the result in all generality, it is easy to demonstrate the result for $p \geq 1$. We have

$$(1) \qquad u' + u = f + g,$$

where $f \ L^p[-\infty,\infty]$, $g \in L^r[-\infty,\infty]$. Thus, u must have the form

$$(2) \qquad u = e^{-t} \int_{-\infty}^{t} f(s)e^s ds + e^{-t} \int_{-\infty}^{t} g(s)e^s ds.$$

The conditions on f and g result in the uniform boundedness of u. Hence $u \in L^p[-\infty,\infty]$ implies that $u \in L^{p+b}[-\infty,\infty]$ for $b > 0$.

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A detailed discussion of inequalities of this type, together
with many references, may be found in the expository paper

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